Math 53: Multivariable Calculus **Sections 102, 103**

Handout for 2020-02-10

Some questions about quadric surfaces

Problem 1. Consider the surface defined by the equation

$$z - xy = 0.$$

What type of quadric surface is this? **Hint:** Apply the linear change of variables $x = u + v$ and $y = u - v$.

Problem 2. Consider the surface defined by the equation

$$\frac{y}{2} - x^2 - z^2 = 0.$$

(a) What is the name of this type of surface? Draw a sketch.
(b) Consider the solid S bounded by this surface and the two planes $y = 8$ and $y = 18$. Write down an expression which computes the surface area of S. Feel free to use a calculator to actually compute it (though I think the integral you get should be reasonable to do by hand as well). **Hint:** Take advantage of the rotational symmetry of S.

Remark. Our current approach to 2(b) is rather ad-hoc and reliant on the rotational symmetry of this particular surface. In Chapter 15 we will learn how to compute surface areas in general.

Problem 3. Consider the surface S defined by the equation

$$y^2 - x^2 + z^2 = 1.$$

(a) What is the name of this type of surface? Draw a sketch.
(b) Let H be the plane $3x - 3y - z + 1 = 0$. This is actually the tangent plane to S at the point $(3, 3, 1)$. (We will learn how to compute tangent planes in Chapter 14.) The intersection of S with H is a pair of intersecting lines. Find vector equations for them.

Remark. 3(b) actually works at every point on the surface S, and the lines arising in this way are the two "rulings" on the quadric surface. This phenomenon is very specific to hyperbolic paraboloids and hyperboloids of one sheet.

Some questions about vector functions

Problem 4. Parametrize the curve of intersection of the two surfaces $z = x^3$ and $y = x^2$.

Problem 5. Find some different surfaces which contain the curve with vector equation $\mathbf{r}(t) = \langle 2t, e^t, e^{2t} \rangle$.

Problem 6. Exhibit the curve $x = \sin t, y = \cos t, z = \sin^2 t$ as the intersection of a circular cylinder with a parabolic cylinder, and use this to help sketch the curve.

Problem 1. By applying the suggested change of variables, the equation becomes

$$z - (u+v)(u-v) = z - u^2 + v^2 = 0$$

which is a hyperbolic paraboloid.

Problem 2.

(a) This is an elliptic paraboloid, "opening" in the positive y direction. I'll omit the sketch; you can check by using your favorite 3D graphing tool. The horizontal traces are actually circles, which will be useful for the next part.

(b) The solid S has three faces: two flat circular faces and a curved one. The trace (of the elliptic paraboloid) at $y = 8$ is the circle $x^2 + z^2 = 2^2$, which has area 4π, and the trace at $y = 18$ is the circle $x^2 + z^2 = 3^2$, which has area 9π.

It remains to consider the curved surface. This part is obtained by rotating the portion of the curve $y = 2x^2$ in the xy-plane with $8 \le y \le 18$ around the y-axis. The inequalities $8 \le y \le 18$ can be replaced with $2 \le x \le 3$, using which we set up the (hopefully familiar) surface area integral:

$$\int_2^3 2\pi x \, \mathrm{d}s = \int_2^3 2\pi x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x. = \int_2^3 2\pi x \sqrt{1 + 16x^2} \, \mathrm{d}x.$$

This integral can be computed by writing $u = 1 + 16x^2$. Its value is

$$-\frac{5}{24}\left(13\sqrt{65} - 29\sqrt{145}\right)\pi \approx 159.96$$

so the final answer is

$$4\pi + 9\pi - \frac{5}{24}\left(13\sqrt{65} - 29\sqrt{145}\right)\pi \approx \boxed{200.80}.$$

Problem 3.

(a) This is a hyperboloid of one sheet, whose "axis" is along the x direction. Again, you can check your sketch by using your favorite 3D graphing tool.

(b) By isolating z in the plane equation and substituting it into the equation of S we get the system

$$\begin{aligned} z &= 3x - 3y + 1 \\ 0 &= 8x^2 - 18xy + 6x + 10y^2 - 6y. \end{aligned}$$

The second equation can be factored:

$$\begin{aligned} 0 &= 8x^2 - 18xy + 6x + 10y^2 - 6y \\ &= 2(4x^2 + (-9y+3)x + (5y^2 - 3y)) \\ &= 2(4x^2 + (-9y+3)x + y(5y-3)) \\ &= 2(4x - (5y-3))(x-y) \\ &= 2(4x - 5y + 3)(x - y). \end{aligned}$$

So either $4x - 5y + 3 = 0$ or $x - y = 0$. In the former case, we get the system

$$\begin{aligned} z &= 3x - 3y + 1 \\ 4x - 5y + 3 &= 0. \end{aligned}$$

If we let $x = t$, then $y = (4t+3)/5$ from the second equation and $z = 3t - 3(4t+3)/5 + 1 = 3t/5 - 4/5$ from the first equation. So a vector equation for this line is

$$\boxed{\mathbf{r}_1(t) = \langle 0, 3/5, -4/5 \rangle + t\langle 1, 4/5, 3/5 \rangle.}$$

In the case $x - y = 0$, we have the system

$$\begin{aligned} z &= 3x - 3y + 1 \\ x - y &= 0. \end{aligned}$$

So proceeding similarly, if we let $x = t$ then $y = t$ from the second equation and $z = 1$ from the first. Hence a vector equation for this line is

$$\boxed{\mathbf{r}_2(t) = \langle 0, 0, 1 \rangle + t\langle 1, 1, 0 \rangle.}$$

As a sanity check, you should verify that these two lines do in fact pass through the point $(3, 3, 1)$.

Problem 4. Take x to be the parameter: $x = t, y = t^2, z = t^3$.

Problem 5. The easiest ones to find are obtained by eliminating the parameter from two of the three components. For example, eliminating from $x = 2t$, $y = e^t$ gives $y = e^{x/2}$. One similarly finds $z = e^x$ and $z = y^2$. There are infinitely many correct answers; these are probably just the simplest to write down.

Problem 6. It's contained in $x^2 + y^2 = 1$ and also in $z = x^2$ (and in $z = 1 - y^2$). It looks like the edge of a potato chip.